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# Approximations of the packet error rate under slow fading in direct and relayed links

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**Abstract:** Most of the existing literature on performance evaluation of fading channels is concentrated on either the symbol error rate of the links, or the outage capacity. We present in this technical report approximation techniques for the packet error rate of quasi-static fading channels. This model applies when the channel coherence time is larger than the signaling time and leads to multiple symbols experiencing the same fading state during a packet transmission. We start by giving a closed-form upper bound on the asymptotic *coding gain* of the packet error rate in quasi-static fading channels. Using the fact that the asymptotic formulation is inversible with respect to the mean signal-to-noise ration (SNR), we present a *packet error outage* metric, with applications to channels where links are subject to both fading and log-normal shadowing effects simultaneously. We then show that a *unit step approximation* can be derived for the packet error rate of block fading channels, that closely matches the numerical computation of the packet error rate in a tractable closed-form expression. Using the asymptotic approximation method, we derive the optimal power allocation for different relaying protocols, especially when optimal combination of the symbols received at the destination may not be possible.

**Key-words:** relay channels, packet error rate, quasi-static fading

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# Approximations du taux d'erreur paquet des liens directs et des canaux à relais dans les canaux à évanouissements quasi-statiques

**Résumé :** Voir résumé anglais

**Mots-clés :** canal à relais, taux d'erreur paquet, évanouissements quasi-statiques

Most of the existing literature on performance evaluation of fading channels is concentrated on either the symbol error rate of the links, or the outage capacity. A good review of these results as well as an interesting framework for the evaluation of symbol error rates in fading channels is available in the book of Simon and Alouini [Simon and Alouini, 2004]. The results presented here root themselves in the work of Wang and Giannakis [Wang and Giannakis, 2003] who presented an asymptotic approximation of symbol error rates in fading channels using a Taylor expansion cut on the first term. The second section presents this framework. This approach is well suited to channels whose probability density function is approximately polynomial near zero, but fails for certain models of fading such as log-normal shadow fading.

The asymptotic approximation of Wang and Giannakis have been extended to general *amplify-and-forward* relays by Ribeiro *et al.* [Ribeiro et al., 2005], who showed that in that case the optimal selection criterion for relays is to maximize the harmonic mean of the source-relay and relay-destination links. Liu *et al.* [Liu et al., 2009] further extended these results and produces a comprehensive treatment of the end-to-end symbol error rates in relay channel for both *amplify-and-forward* and *decode-and-forward* protocols, including asymptotic approximations, and for a variety of modulation schemes [See in particular Liu et al., 2009, Ch.5]. Using the results from Zheng and Tse [Zheng and Tse, 2003], the study of symbol error rates at high SNR can be related to the study of the capacity outage probability at high SNR[1]. Related works thus include asymptotic approximations of the outage probability in fading channels. In a manner similar to the approach we present here, Annavajjala *et al.* treated the asymptotical outage probability of direct links as well as relayed *amplify-and-forward* and *decode-and-forward* protocols [Annavajjala et al., 2007].

These results are focused on the symbol error rate of fading channels ; when the fading is relatively fast compared to the symbol transmission duration, with proper interleaving one can extend them to packet error rates. On the other hand, when the fading is much slower than the symbol transmission time one has to consider that most symbols in the packet will experience the same fading state – a model known as *block fading* or *quasi-static* fading. Wang and Giannakis' approach has been extended recently by Xi *et al.* for the packet error rate of block fading channels [Xi et al., 2010], including the asymptotic performance of *decode-and-forward* in an ideal relay channel.

The work presented in this paper is based on the results of Xi *et al.* We strengthen their results by giving a closed-form approximation of the asymptotic *coding gain* rather than a numerical evaluation, for two usual forms of bit error rate expressions used in realistic cases. Using the fact that the asymptotic formulation is inversible with respect to the mean signal-to-noise ration (SNR), we present a *packet error outage* metric, with applications to channels where links are subject to both fading and log-normal shadowing effects simultaneously. We then show that a *unit step approximation* can be derived for the packet error rate of block fading channels, that closely matches the numerical computation of the packet error rate in a tractable closed-form expression. Using the asymptotic approximation method, we derive the optimal power allocation for different relaying protocols, especially when optimal combination of the symbols received at the destination may not be possible. We then present immediate work perspectives of interest based on our results.

[1] Lemma 5 in Zheng and Tse's paper states that at high SNR, a detection error is very likely, conditioned on a channel outage

# 1 System model

In communication systems, fading effects corrupt the amplitude of the envelope of the received signals. The book by Tse and Vishwanath [Tse and Vishwanath, 2008] gives a good and concise overview of the basic channel models, and the derivation of the classical *discrete baseband model* from the general continuous multipath fading channel model [See Tse and Vishwanath, 2008, Ch.2]. We use their notation here and consider a single-tap discrete complex baseband channel model where the signal $y[m]$ received at time $m$ depends on the sent signal $x[m]$, an additive white complex gaussian noise term $w[m] \sim \mathcal{CN}(0, N_0)$ and an aggregate tap gain $h[m]$:

$$y[m] = h[m]x[m] + w[m] \tag{1}$$

In practice, most performance metrics for communication systems are based on the SNR of the received symbols. Assuming that symbols are sent with an average power $P$ – with equality for phase-shift keyed modulations – the *instantaneous SNR* of the received symbols is :

$$\gamma[m] = \frac{|h[m]|^2 P}{N_0} \tag{2}$$

The *mean SNR* may be computed as $\bar{\gamma} = \mathbb{E}\left[|h[m]|^2\right]\frac{P}{N_0}$. The effect of fading channels is captured through the probability distribution of the squared aggregate tap gain $|h[m]|^2$, and the usual models we use in this paper may be found in [Tse and Vishwanath, 2008, Ch.2] or [Simon and Alouini, 2004, Ch.3]. The probability density functions (p.d.f.) for these models are summarized in Tab.1.

| | p.d.f. of $\|h[m]\|^2$ | Parameters |
|---|---|---|
| Rayleigh Model | $\frac{1}{\bar{\gamma}}\exp\left(\frac{\gamma}{\bar{\gamma}}\right)$ | $\bar{\gamma}$ |
| Rice Model | $\frac{(1+K)e^{-K}}{\bar{\gamma}}\exp\left(\frac{-\gamma(1+K)}{\bar{\gamma}}\right) I_0\left(2\sqrt{K(K+1)\frac{\gamma}{\bar{\gamma}}}\right)$ | $K, \bar{\gamma}$ |
| Nakagami model | $\frac{m^m \gamma^{m-1}}{(\bar{\gamma})^m \Gamma(m)}\exp\left(-\frac{m\gamma}{\bar{\gamma}}\right)$ | $m, \bar{\gamma}$ |

Table 1: Probability density functions for the fading models considered in this paper. $I_0(\cdot)$ is the Bessel function of type 1 and order 0, $\Gamma(\cdot)$ is the Gamma function.

The metric of interest in this paper is based on the instantaneous symbol error rate $p_s(\gamma)$, which is dependent on the modulation used and will represent the probability of a symbol detection error at the given SNR. When the knowledge of the received instantaneous SNR is known only statistically through the mean SNR and the probability distribution of $|h[m]|^2$, we can compute the mean symbol error rate of the fading channel as follow as the expectation of the instantaneous symbol error rate over the fading channel p.d.f. $f_\gamma(\gamma)$, as given in Tab.1:

$$\bar{p}_s(\bar{\gamma}) = \int_0^\infty p_s(\gamma) f_\gamma(\gamma) d\gamma \tag{3}$$

We consider as a metric in our work the *packet error rate* (PER), where packets are formed with $N$ transmitted symbols. Without any coding on this packet, the probability of a packet error at a given instantaneous SNR is given by:

$$p_p(\gamma) = 1 - (1 - p_s(\gamma))^N \tag{4}$$

In fast fading channels, we may use interleaving techniques and let sent symbols from a single packet experience different fading states, leading the mean packet error rate to be based on the mean symbol error rate through natural averaging:

$$\bar{p}_{p,\text{fast}}(\bar{\gamma}) = 1 - (1 - \bar{p}_s(\bar{\gamma}))^N \tag{5}$$

We will call this metric the *ergodic packet error rate*. On the other hand, when the fading is slow, symbols in a packet will experience the same or similar fading states. As with the mean symbol error rate (3), we thus have to integrate the instantaneous packet error rate (4) over the probability distribution of $|h[m]|^2$ to get the *block packet error rate*, which is our metric of interest in this paper:

$$\bar{p}_{p,\text{slow}}(\bar{\gamma}) = \int_0^\infty p_p(\gamma) f_\gamma(\gamma) d\gamma = 1 - \int_0^\infty (1 - p_s(\gamma))^N f_\gamma(\gamma) d\gamma \tag{6}$$

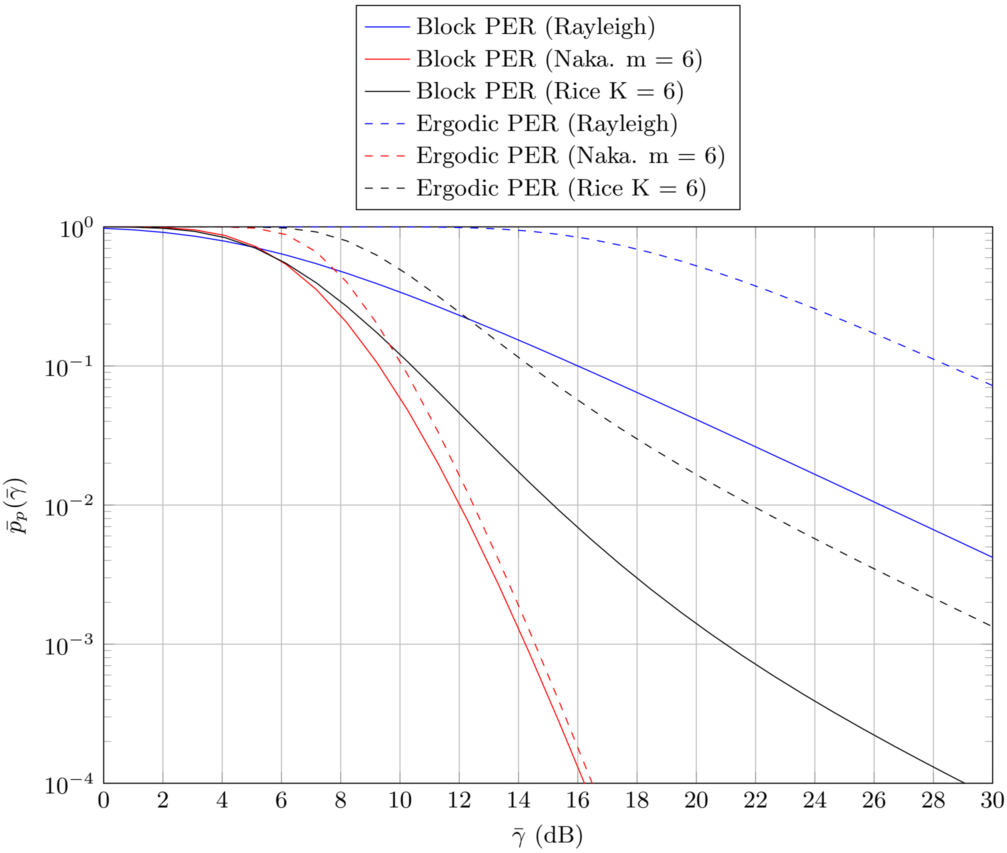



Figure 1: Comparison between the ergodic PER and block PER.

## 2 Approximations of the packet error rate in single links

As an illustrative exemple, Fig.1 compares the block packet error rate (PER) and ergodic PER for various fading channel models. As we can see on the figure, the ergodic PER seems to upper bound the block PER in every case. The theorem below generalizes this result.

**Theorem 1.** *The ergodic packet error rate as defined in* (5) *is an upper bound to the block packet error rate as defined in* (6)*.*

*Proof.* We can write $\bar{p}_{p,\text{slow}}(\bar{\gamma})$ and $\bar{p}_{p,\text{fast}}(\bar{\gamma})$ as expectations of a function of $\gamma$:

$$\bar{p}_{p,\text{slow}}(\bar{\gamma}) = 1 - \mathbb{E}\left[(1-p_s(\gamma))^N\right] \tag{7a}$$

$$\bar{p}_{p,\text{fast}}(\bar{\gamma}) = 1 - (1 - \mathbb{E}\left[p_s(\gamma)\right])^N \tag{7b}$$

Noting that for $x \in (0,1)$, $x \mapsto (1-x)^N$ is convex for any positive N, by Jensen's inequality we have that:

$$\mathbb{E}\left[(1-p_s(\gamma))^N\right] \geq (1 - \mathbb{E}\left[p_s(\gamma)\right])^N \tag{8}$$

Identifying (7) in (8) concludes the proof. □

In this work, we restrict ourselves to symbol error rates represented by the generic functions of the following forms[2]:

$$p_{s,\text{th}}(\gamma) = Q(\sqrt{k\gamma}) \tag{9}$$

$$p_{s,\text{fit}}(\gamma) = \frac{1}{2}\exp(-\gamma^{\alpha}) \tag{10}$$

The first function is the theoretical symbol error rate of PSK modulations, whereas the second one is classically used to fit bit or symbol error rates in realistic systems.

## 2.1 Asymptotic approximations

The main result by Wang and Giannakis [2003, Prop.1] is to show that integrals of the form of (3) may be approximated at high SNR, when $\bar{\gamma} \to \infty$ by:

$$a \int_0^{\infty} \gamma^t p_s(\gamma) d\gamma \cdot \bar{\gamma}^{-(t+1)} \tag{11}$$

This approximation requires some conditions on the fading p.d.f., the main one being that this

| Model | $t$ | $a$ |
|---|---|---|
| Rayleigh | 0 | 1 |
| Rice | 0 | $(1+K)e^{-K}$ |
| Nakagami | $m-1$ | $m^m/\Gamma(m)$ |

Table 2: Parameters $a$ and $t$ for channels of interest.

p.d.f. may be expanded as a single polynomial term when $\gamma \to 0$ [See Wang and Giannakis, 2003, Sec.II]. The parameters $a$ and $t$ are dependent on the fading channel model only and are listed in Tab.2 for the models we consider in the paper. This leads in many cases to an approximation of the form:

$$\bar{p}_s(\bar{\gamma}) \approx G_c \bar{\gamma}^{-G_d} \tag{12}$$

where $G_c$ is termed the *coding gain* and $G_d$ is the *diversity gain*. Xi et al. [2010] showed that the result is applicable to the block PER (6), replacing the term $p_s(\gamma)$ in (11) by $p_p(\gamma)$ from (4). The integral of interest is thus, for a given packet size $N$ and a parameter $t > -1$:

$$I = \int_0^{\infty} \gamma^t (1 - (1-p_s(\gamma))^N) d\gamma \tag{13}$$

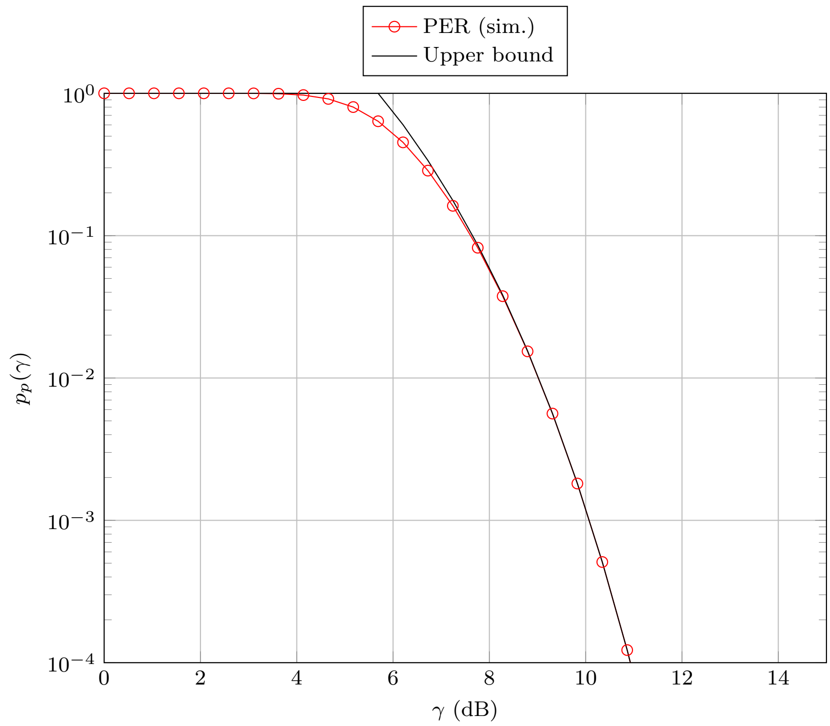



Figure 2: Upper bound on the instantaneous PER.

We may approximate at this point the instantaneous PER by an upper bound. We know, by a probability union bound, that for any $\gamma \in (0, \infty)$, $(1 - (1 - p_s(\gamma))^N) \leq N p_s(\gamma)$. The bound gets tighter as $p_s(\gamma)$ decreases when $\gamma \to \infty$. We can thus construct a relatively close upper bound on the instantaneous packet error rate, as shown on Fig.2:

$$p_p(\gamma) \leq \min\{1, N p_s(\gamma)\} \tag{14}$$

Using this bound, we have:

$$I \leq \frac{1}{t+1}\gamma^{*(t+1)} + N \int_{\gamma^*}^{\infty} \gamma^t p_s(\gamma) d\gamma \tag{15}$$

In this expression, $\gamma^*$ is the solution to the equation $N p_s(\gamma^*) = 1$ and only depends on the block size and modulation. For the symbol error rates (9) and (10), it can be expressed as:

$$\gamma^*_{\text{th}} = \frac{2}{k}\left(\operatorname{erfc}^{-1}\left(\frac{2}{N}\right)\right)^2 \qquad \gamma^*_{\text{fit}} = \left(-\log\left(\frac{2}{N}\right)\right)^{\frac{1}{\alpha}} \tag{16}$$

To integrate the second term in (15), we will make use of the following lemma.

[2] $Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} \exp\left(-\frac{u^2}{2}\right) du$ is the tail probability of a standard normal distribution.

**Lemma 1.** *Let $\Gamma(s,x)$ be the incomplete upper gamma function*[3]*. We have the relation:*

$$\int x^t \Gamma(s,x)\, dx = \frac{x^{t+1}}{t+1}\Gamma(s,x) - \frac{1}{t+1}\Gamma(t+s+1,x)$$

*Proof.* We proceed using integration by parts. We have:

$$\int x^t \Gamma(s,x)\, dx = \frac{x^{t+1}}{t+1}\Gamma(s,x) - \frac{1}{t+1}\int x^{t+1}\Gamma'(s,x)dx$$

From Olver et al. [2010, Eq.8.8.14], we know that $\Gamma'(s,x) = -x^{s-1}e^{-x}$. Thus:

$$\frac{1}{t+1}\int x^{t+1}\Gamma'(s,x)dx = \frac{1}{t+1}\int -x^{t+s}e^{-x}dx$$

Identifying $-x^{t+s}e^{-x} = \Gamma'(t+s+1,x)$ completes the proof. □

Using the relation between the $Q$ function and the incomplete gamma function[4] from Olver et al. [2010, Eq.7.11.2], we can express (15) and thus an upper bound on the coding gain $G_c^{(\text{block,th})}$ of theoretical PSK modulations in block fading channels using the following proposition, with $N$ the block size and $a, t$ from Tab.2.

**Proposition 1.** *The coding gain $G_c^{(\text{block,th})}$ of theoretical PSK modulations in block fading channels is bounded above by:*

$$G_c^{(\text{bloc,th})} \leq \left(\frac{2}{k}\right)^{t+1} \frac{aN}{2(t+1)\sqrt{\pi}}\Gamma\left(t+\frac{3}{2}, \frac{k\gamma^*}{2}\right)$$

*Proof.* Using Olver et al. [2010, Eq.7.11.2], we can write:

$$\int_{\gamma^*}^{\infty} \gamma^t p_s(\gamma)d\gamma = \frac{1}{2\sqrt{\pi}}\int_{\gamma^*}^{\infty}\gamma^t\Gamma\left(\frac{1}{2}, \frac{k\gamma}{2}\right)d\gamma$$

From Lemma.1, with a variable change $u = k\gamma/2$, we have:

$$\begin{aligned}
&\frac{1}{2\sqrt{\pi}}\int_{\gamma^*}^{\infty}\gamma^t\Gamma\left(\frac{1}{2}, \frac{k\gamma}{2}\right)d\gamma \\
&= \left(\frac{2}{k}\right)^{t+1}\frac{1}{2\sqrt{\pi}}\int_{\frac{k\gamma^*}{2}}^{\infty} u^t\Gamma\left(\frac{1}{2}, u\right)du \\
&= \left(\frac{2}{k}\right)^{t+1}\frac{1}{2(t+1)\sqrt{\pi}}\left[u^{(t+1)}\Gamma\left(\frac{1}{2}, u\right) - \Gamma\left(t+\frac{3}{2}, u\right)\right]_{\frac{k\gamma^*}{2}}^{\infty}
\end{aligned}$$

The asymptotic expansion of $\Gamma(s,x)$ is [Olver et al., 2010]:

$$\Gamma(s,x) \sim x^{s-1}e^{-x}\sum \frac{\Gamma(s)}{\Gamma(s-k)}x^{-k}$$

[3] $\Gamma(s,x) = \int_x^{\infty} t^{s-1}e^{-t}dt$
[See Olver et al., 2010, Ch.8]

[4] $Q(x) = \frac{1}{2\sqrt{\pi}}\Gamma\left(\frac{1}{2}, \frac{x^2}{2}\right)$

The exponential will dominate any polynomial term when $x \to \infty$. We thus have:

$$\lim_{u\to\infty} u^{(t+1)}\Gamma\left(\frac{1}{2}, u\right) - \Gamma\left(t+\frac{3}{2}, u\right) = 0$$

The integral $I$ from (13) can be bound by:

$$I \leq \frac{1}{t+1}\gamma^{*(t+1)} - \left(\frac{2}{k}\right)^{t+1}\left(\frac{k}{2}\right)^{t+1}\frac{N}{2(t+1)\sqrt{\pi}}\gamma^{*(t+1)}\Gamma\left(\frac{1}{2}, \frac{k\gamma^*}{2}\right) + \left(\frac{2}{k}\right)^{t+1}\frac{N}{2(t+1)\sqrt{\pi}}\Gamma\left(t+\frac{3}{2}, \frac{k\gamma^*}{2}\right)$$

The first two terms cancel out since $(2\sqrt{\pi})^{-1}\Gamma(1/2, k\gamma^*/2) = 1/N$ by definition of $\gamma^*$. Reinjecting the remaining term in (11) gives the proposition. □

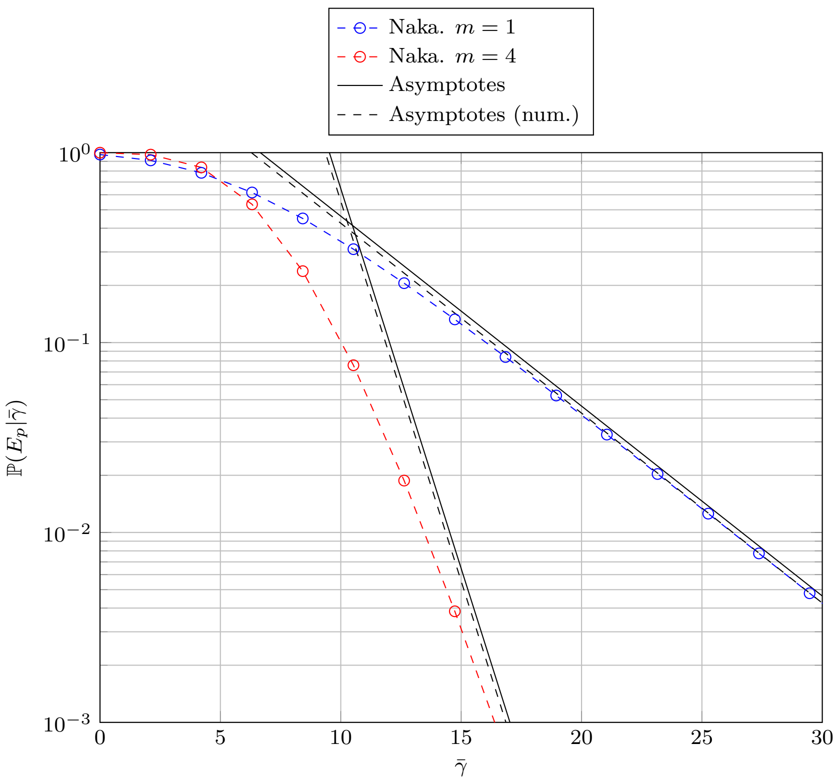



Figure 3: Asymptotic approximation of the PER, using a Nakagami channel model, BPSK modulation and a packet size of 312 bits.

We compare the upper bound on the coding gain given in Prop.1 with a numerical computation of the coding gain on Fig.3. We consider for this application a binary phase-shift keying (BPSK) modulation scheme for which $k = 2$ in (9). The asymptotes on the PER given by a numerical computation of the coding gain and the ones using the bound of Prop.1 are very close to one another. On Fig.1, when $m = 1$, the bound is offset from the real asymptote by less that

0.3 dB, and by 0.1 dB when $m = 4$. We have further been able to assess empirically that the bound gets tighter as $N$ or $t$ increases.

A similar bound for symbol error rates of the form (10) may be derived in a simple way along the line of the proof of Prop.1, and we have the following proposition:

**Proposition 2.** *The coding gain $G_c^{(\text{block,fit})}$, associated with empirical symbol error rates of the form* (10) *in block fading channels, is bounded above by:*

$$G_c^{(\text{block,fit})} = \frac{a}{t+1}\gamma^{*(t+1)} + \frac{aN}{2\alpha}\Gamma\left(\frac{t+1}{\alpha}, \gamma^{*\alpha}\right)$$

## 2.2 Packet error outage

In some cases, fading effects are not enough to model the propagation environment of a system. In vehicular networks, cellular mobility models or body area networks[5], the mean SNR $\bar{\gamma}$ may be subject to variation over time, leading to a composite model. This effect is called a *shadowing* effect and is due to the masking of line of sight waves by buildings, or more generally by a macroscopical change in the propagation environment. In most applications, the shadowing effects are much slower than the transmission duration of a packet, and are modeled by considering that the mean SNR is a log-normal random variable and follows a $\mathcal{LN}(\mu_0, \sigma_S)$ probability distribution, where $\mu_0$ is the global mean SNR and $\sigma_S$ the variance of $\bar{\gamma}$ around the mean when expressed in dB. When the system knows, or can predict, the value of $\bar{\gamma}$ at the time of transmission, the block PER (6) is a metric of choice for performance evaluation and parameter optimization. On the other hand, if only the global mean $\mu_0$ is known, it is very difficult to express the probability distribution of the instantaneous SNR $\gamma$, and thus to derive the packet error rate. An approach to lift this limitation is to consider the block *packet error outage* (PEO) as the probability that the block PER will rise beyond a given threshold $P^*$. The PEO can thus be written as:

$$p_o(P^*) = \mathbb{P}\left\{\bar{p}(\bar{\gamma}) \geq P^*\right\} \tag{17}$$

Using asymptotic approximations as presented in the preceding section, one can actually use invert $\bar{p}(\bar{\gamma})$. Since we can reasonably assume that the mean PER is nonincreasing and continuous, we can write the PEO as follows:

$$p_o(P^*) = \mathbb{P}\left\{\bar{\gamma} \leq \bar{p}^{-1}(P^*)\right\} \tag{18}$$

When the shadowing is modeled as a log-normal random variable, expressing the mean SNR in decibels allows to derive the PEO using the cumulative distribution function (c.d.f.) of a normal random variable $\Phi(\cdot)$[6]. Considering that the global mean received SNR is $\mu_0$, and taking care of converting $\bar{\gamma}^* = \bar{p}_p^{-1}(P^*)$ in decibels, the PEO is thus written as:

$$p_o(P^*) = \Phi\left(\frac{\bar{\gamma}^* - \mu_0}{\sigma_S}\right) \tag{19}$$

Fig.4 represents the PEO for a Rayleigh fading channel using a threshold for the PER at 1%. The PEO curves define the required mean *received* SNR to ensure a 1% PER at the receiver in a given percentage of the shadowing states. Intuitively, the higher the variance of the shadowing effect the higher the required received SNR to fill this condition, which is in agreement with Fig.4. Furthermore, changing the required PER threshold $P^*$ will not change the shape of the PEO curves, but only translate them along the $\mu_0$ axis.

[5] Missing citations

[6] $\Phi(x) = \frac{1}{2}\left(1 + \text{erf}\,(x)\right)$ is the c.d.f. for a centered Gaussian random variable with variance 1.

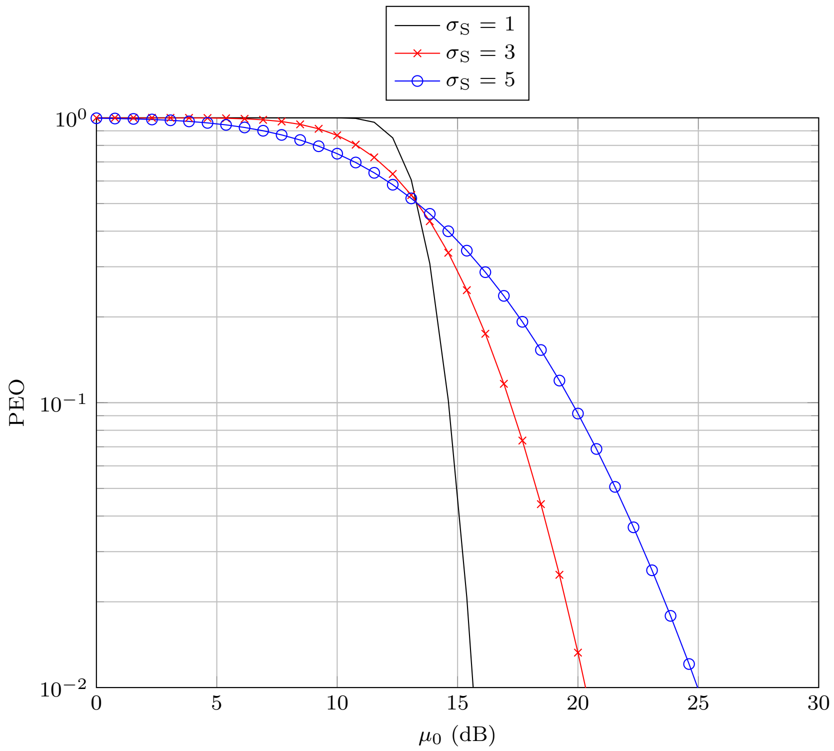



Figure 4: Packet error outage for a Rayleigh block fading channel, with a PER threshold $P^* = 0.01$.

## 2.3 Unit step approximations

As mentioned in Wang and Giannakis [2003] and seen on Fig.5, block fading channels represented by a Rician distribution are not well approximated by asymptotes. On Fig.5, we represent the *equivalent* Nakagami channel obtained by equating the *Amount of Fading* (AF) defined in Simon and Alouini [2004, Sec.2.2], a metric computed from the mean and variance of fading distributions. We can see that both channels have a similar behavior for low values of $\bar{\gamma}$, but the Rician fading model generates an asymptote in $O(\bar{\gamma}^{-1})$, whereas the Nakagami fading model's asymptote is $O(\bar{\gamma}^{-m})$ (see Tab.2 and (11)). Using the simple asymptote formulations, it is possible to extract the crossing between the asymptotes and therefore define a piecewise function approximation the PER of a Rician block fading channel better at medium SNR. However, this method is not entirely satisfactory. In coded packet schemes, a common approach is to consider the instantaneous SER or PER as a unit step function whose value is 1 below some threshold SNR $\gamma_{\text{th}}$ and 0 beyond:

$$\tilde{p}(\gamma) = \begin{cases} 1, & \text{if } \gamma \leq \gamma_{\text{th}} \\ 0, & \text{otherwise} \end{cases} \tag{20}$$

This approach has been show to be particularly efficient in turbo-coded schemes that are iteratively decoded [El Gamal and Hammons, 2001, Chatzigeorgiou et al., 2008], when the threshold is set to the *waterfall* threshold of the decoding scheme – the SNR beyond which the the decoding

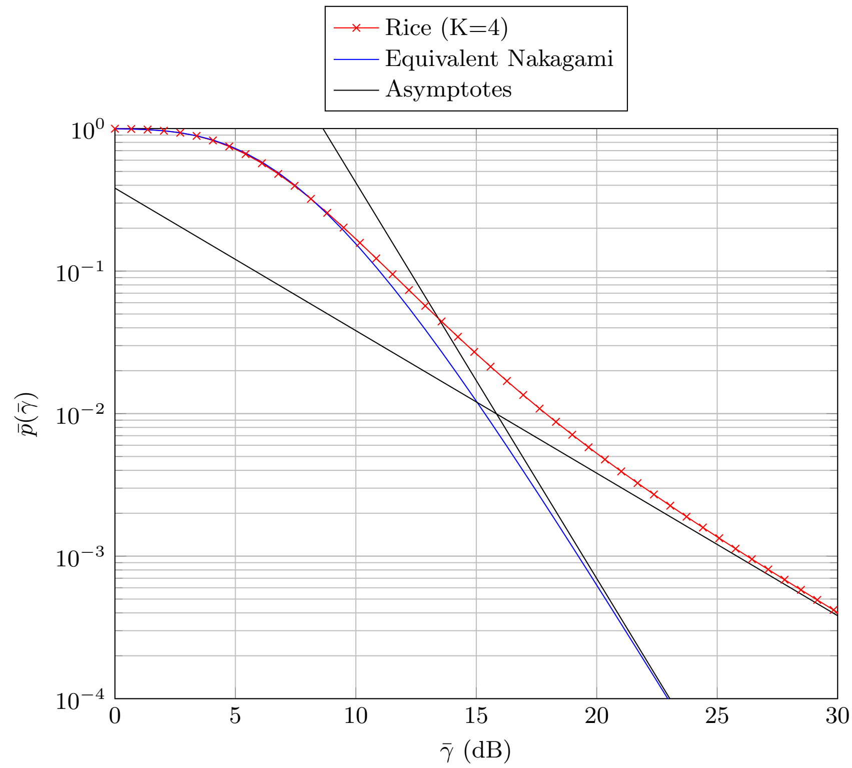



Figure 5: PER of a Rician fading channel with its equivalent Nakagami fading channel.

algorithm provably converges. In the uncoded case, we can set this threshold to the solution of:

$$(1 - p_s(\gamma_{\text{th}}))^N = \frac{1}{2} \tag{21}$$

For the functions (9) and (10), closed-form solutions may be obtained for $\gamma_{\text{th}}$. The obvious interest of this approximation is that the resulting formulation is based on the c.d.f. $F_\gamma(\gamma)$ relative to the channel block fading p.d.f. $f_\gamma(\gamma)$. From (6) and using the unit step approximation $\tilde{p}(\gamma)$ for the instantaneous PER, the approximation is derived as[7]:

$$\bar{p}_p(\bar{\gamma}) = \int_0^{\gamma_{\text{th}}} f_\gamma(\gamma) d\gamma = F(\gamma_{\text{th}}) \tag{22}$$

This approximation is mathematically valid whenever the c.d.f. exists, and can thus provide a way to treat cases where the asymptotic approximation is looser than expected, e.g. for the Rician fading model. It can also be of use when the asymptotic approximation does not exist, which is the case for the log-normal fading model whose p.d.f. behaves exponentially near 0 [Recall the conditions in Wang and Giannakis, 2003, Sec.II]. As seen on Tab.3, the closed form expressions are more tractable than their integral counterparts, and Fig.6 indicates that the approximation is quite close for the channel parameters displayed. The approximation loosens for lower packet sizes, i.e. for low values of $N$, but gets tighter as $N$ increases. An intuitive explanation is that when $N$ increases the slope of the instantaneous PER also increases and gets

[7] $f_\gamma(\gamma)$ and $F_\gamma(\gamma)$ are implicitely dependent on $\bar{\gamma}$

| Model | Block PER approx. |
|---|---|
| Rayleigh | $1 - \exp\left(-\frac{\gamma_{\mathrm{th}}}{\bar{\gamma}}\right)$ |
| Rice | $1 - Q_1\left(\sqrt{2K}, \sqrt{2(K+1)\frac{\gamma_{\mathrm{th}}}{\bar{\gamma}}}\right)$ |
| Nakagami | $\frac{1}{\Gamma(m)}\gamma(m, \frac{m\gamma_{\mathrm{th}}}{\bar{\gamma}})$ |
| Log-Normal | $Q\left(\frac{\bar{\gamma} - 10\log_{10}(\gamma_{\mathrm{th}})}{\sigma_S}\right)$ |

Table 3: Unit step approximations of the block fading PER for usual channel models.

closer to a unit step function. The main issue for these results is that, contrary to the asymptotic approximations, the unit step approximations are not always invertible and thus do not yield closed-form power allocations for a given target PER. They also require numerical root-finding algorithms to be used in the PEO metric of the preceding section. Although it seems that the unit step approximation lower bounds the block PER in the Fig.6, we were not able to prove it and it must depend on the shape of the original instantaneous PER relative to the unit step function and the p.d.f. of the considered fading channel model.

# 3 Approximations of the packet error rate in relayed links

The elementary building block of cooperative networks is the relay channel (Fig.7) where the source transmits towards the destination with the help of a relay. In this section, we study the end-to-end PER of relay channels under block Rayleigh or Rician fading channel models, and we use the asymptotic approximations presented in the preceding section to derive asymptotically optimal power allocations in these setups. We consider that the relay node is a wireless node, and cannot send and receive information at the same time, leading to a *half-duplex* mode of operation. We identify 3 possible behaviors for the relay channel (Tab.4), depending on wether the destination listens to the transmission of both the source and relay and tries to combine the received signals. The total cooperation behavior has been treated by Xi et al. [2010] who derived the asymptotic approximation of the end-to-end block PER. In many practical cases however, due to hardware restrictions or limited signal processing capabilities, the destination will be unable to optimally combine the received signals and obtain the gains mentioned in Xi et al. [2010], thus justifying the study of less capable cooperation models.

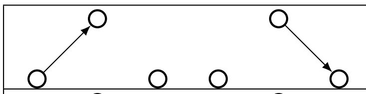
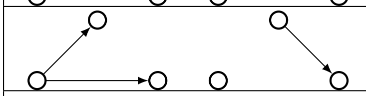
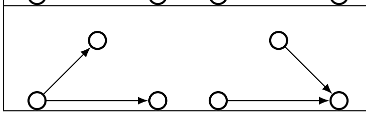
| | |
|---|---|
|  | Multihop transmission. The direct path between the source and destination is not used. |
|  | Partial cooperation, where the destination receives the data directly from the source or from the relay. |
|  | Complete cooperation with optimal combining at the destination. The source and relay transmit concurrently in the second phase. |

Table 4: Relaying schemes considered in this paper.

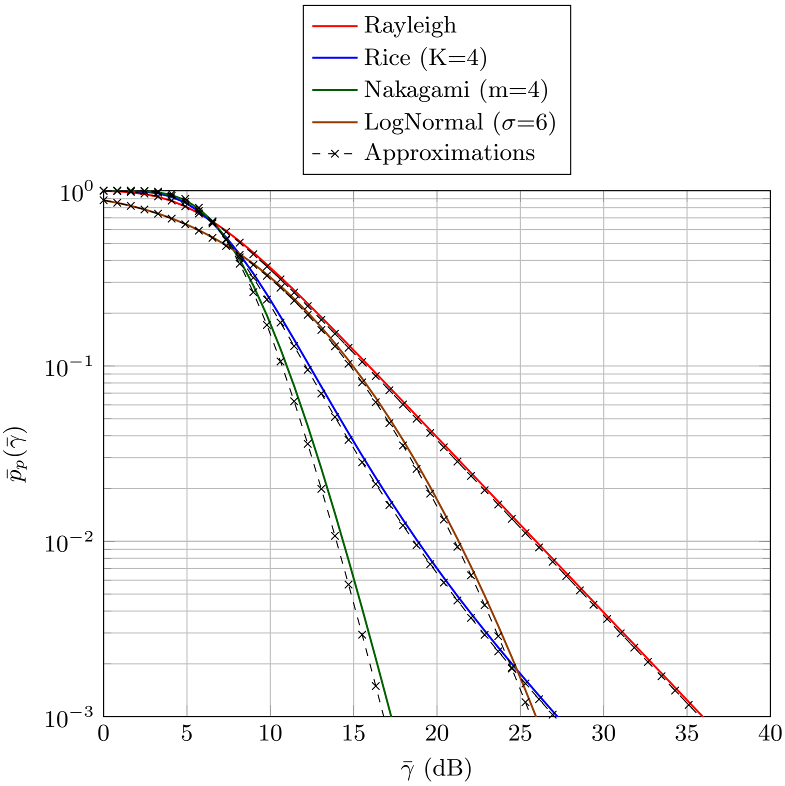



Figure 6: Unit step approximations for the block PER of the fading channel models presented in Tab.3, with theoretical BSPK modulation (see (9) with $k = 1$) and a packet size $N = 312$ bits.

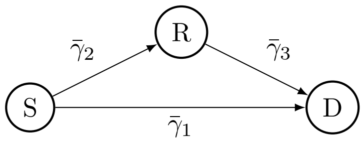



Figure 7: Relay channel, represented with the mean received SNR of each link.

As represented on Fig.7, the mean SNR is different for the 3 links in the relay channel. From the received SNR at the relay from the source $\bar{\gamma}_1$, we can extract the pathloss coefficient of the $S \rightarrow D$ link in the following manner, with $\bar{\gamma}_S = P_S/N_0$ the transmission power of the source normalized over the noise power:

$$s_1 = \frac{\bar{\gamma}_1}{\bar{\gamma}_S} \tag{23}$$

In a similar manner, we can derive $s_2$ and $s_3$, the pathloss coefficients of the $S \rightarrow R$ and $R \rightarrow D$ links respectively. We consider here that the nodes have knowledge of these pathloss coefficients and that in particular, the source and relay nodes are able to adapt their transmission powers with respect to this knowledge. We aim at allocating a global power $P_{\text{tot}}$ between the source and the relay – or equivalently a global SNR $\bar{\gamma}_{\text{tot}}$. We can introduce a power sharing term $\delta$ defined such that:

$$\bar{\gamma}_S = \delta \bar{\gamma}_{\text{tot}} \qquad \bar{\gamma}_R = (1 - \delta)\bar{\gamma}_{\text{tot}} \tag{24}$$

For the first model of Tab.4, an end-to-end error occurs if the $S \to R$ link fails, or if the $R \to D$ fails while the $S \to R$ link succeeds. The end-to-end error probability can thus be written:

$$P_1 = \bar{p}_p(\bar{\gamma}_2) - (1 - \bar{p}_p(\bar{\gamma}_2))\bar{p}_p(\bar{\gamma}_3) \tag{25}$$

For the second model of Tab.4, an error occurs if both paths are in error, and we thus have the end-to-end probability:

$$P_2 = \bar{p}_p(\bar{\gamma}_1) \left[\bar{p}_p(\bar{\gamma}_2) - (1 - \bar{p}_p(\bar{\gamma}_2))\bar{p}_p(\bar{\gamma}_3)\right] \tag{26}$$

Since we consider Rayleigh and Rician block fading models, the functions $\bar{p}_p(\bar{\gamma}_i)$ may be asymptotically approximated as follows, with $G_i$ the coding gain associated with the link $i$:

$$\bar{p}_p(\bar{\gamma}_i) \approx \frac{G_i}{\bar{\gamma}_i} \tag{27}$$

Furthermore, the single link functions $\bar{p}_p(\bar{\gamma}_i)$ are monotonously decreasing. At high SNR, the product of two of these functions will quickly become negligible relatively to the other terms in the end-to-end PER (25) and (26). After some algebraic manipulations, we can thus express the asymptotic approximations of the end-to-end PER in both these models as:

$$P_1 \approx \frac{1}{\bar{\gamma}_{\text{tot}}} \left( \frac{G_2}{\delta s_2} + \frac{G_3}{(1-\delta)s_3} \right) \tag{28}$$

$$P_2 \approx \frac{1}{\bar{\gamma}_{\text{tot}}^2} \frac{G_1}{s_1} \left( \frac{G_2}{\delta^2 s_2} + \frac{G_3}{(1-\delta)\delta s_3} \right) \tag{29}$$

The derivation of the end-to-end PER of the third model of Tab.4 is described in Xi et al. [2010], and can be written as follows, with $G_{\text{MRC}}$ the evaluation of (13) for $t = 1$:

$$P_3 \approx \frac{1}{\bar{\gamma}_{\text{tot}}^2} \frac{1}{s_1} \left( \frac{G_1 G_2}{\delta s_2} + \frac{G_{\text{MRC}}}{(1-\delta)\delta s_3} \right) \tag{30}$$

We can immediately see that for both the second and third model, the end-to-end probability is $O(\bar{\gamma}^{-2})$ which means that in those relaying modes the diversity gain of using a relay is equal to 2. The asymptotically optimal power allocations $\delta_i$ for each of these models can be deduced from these equations, and we have the following proposition:

**Proposition 3.** *Consider a relay channel as represented on Fig.7. For the first model of Tab.4, the asymptotically optimal power allocation $\delta_1$ is written:*

$$\delta_1 = \frac{\sqrt{\beta_2}}{\sqrt{\beta_2} + \sqrt{\beta_3}} \tag{31}$$

*For the second model:*

$$\delta_2 = \begin{cases} \frac{\beta_2 - 4\beta_3 + \sqrt{\beta_2(\beta_2 + 8\beta_3)}}{4(\beta_2 - \beta_3)}, & \text{if } \beta_2 \neq \beta_3 \\ \frac{2}{3}, & \text{otherwise} \end{cases} \tag{32}$$

*For the third model:*

$$\delta_3 = 1 + \frac{\beta_2'}{\beta_3'} - \sqrt{\left(\frac{\beta_2'}{\beta_3'}\right)^2 + \frac{\beta_2'}{\beta_3'}} \tag{33}$$

*with:*

$$\beta_1 = \frac{s_1}{G_1} \quad \beta_2 = \frac{s_2}{G_2} \quad \beta_3 = \frac{s_3}{G_3} \quad \beta_2' = \frac{s_2}{G_1 G_2} \quad \beta_3' = \frac{s_3}{G_{MRC}} \tag{34}$$

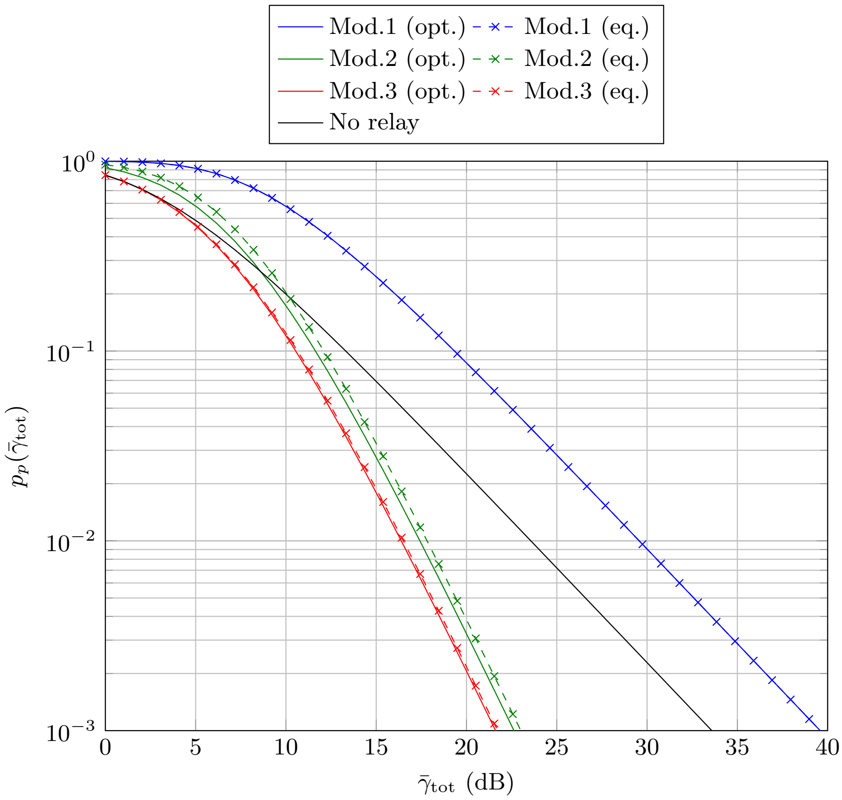



Figure 8: Comparison between the asymptotically optimal power allocation and an equal power allocation between the source and relay node. Here, $s_1 = s_2 = s_3 = 0$ dB.

*Proof.* For the first model, deriving the equation (28) w.r.t. $\delta$ leads to the conclusion that the asymptotically optimal $\delta_1$ is a solution of:

$$\delta^2\beta_2 - (1-\delta^2)\beta_3 = 0$$

This polynomial has a unique root located in $(0, 1)$. In a similar manner, when $\beta_2 \neq \beta_3$, the asymptotically optimal $\delta_2$ is a solution of:

$$2(1-\delta)^2\beta_3 + \beta_2\delta(1-\delta) = 0$$

The third model follows along the same lines and is treated in Xi et al. [2010]. □

For every model considered, we can readily see that the asymptotically optimal power allocation is only a function of $s_2$ and $s_3$ and the coding gains, but is not related to the quality of the $S \to D$ link, nor is it related to the power to allocate $\bar{\gamma}_{\text{tot}}$. We draw a comparison between these asymptotically optimal power allocations and an equal power allocation between the source and the relay on Fig.8 and Fig.9. As predicted, the diversity gain of the second and third cooperation mode in Tab.4 is clearly seen on the slope of their relative end-to-end PER. The first model may in fact provide a coding gain only compared to a direct non-cooperative transmission, if both the $S \to R$ and $R \to D$ link are better than the $S \to D$ link. We can also validate that since the diversity gain is not dependent on the power allocation method, optimizing over the coding gain of the end-to-end PER is a valid approach over the whole SNR range.

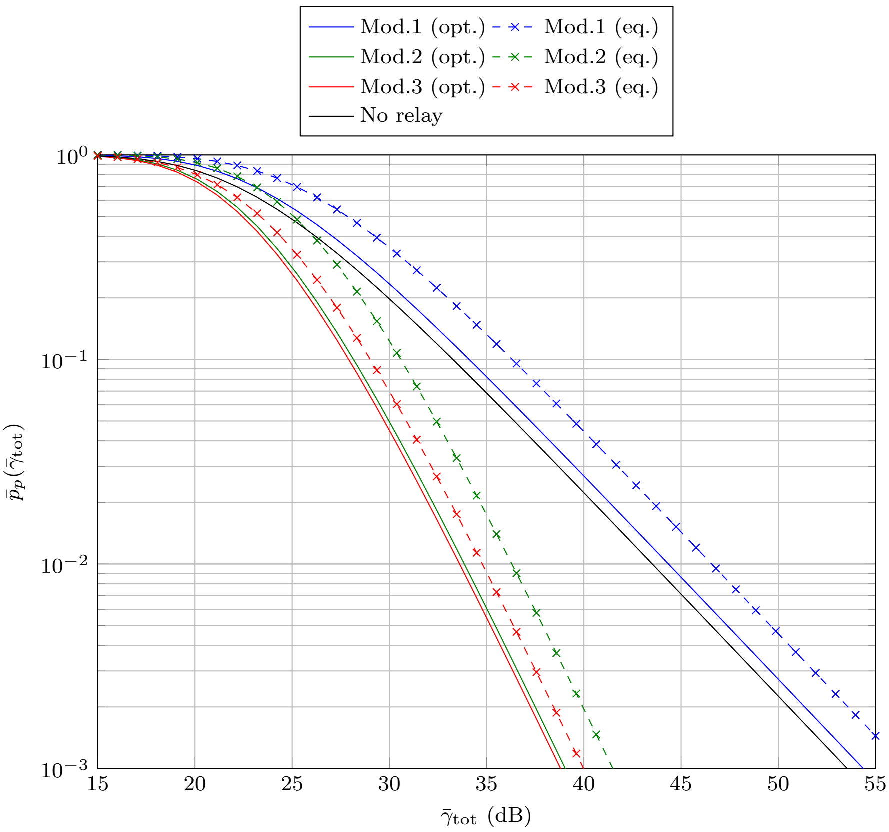



Figure 9: Comparison between the asymptotically optimal power allocation and an equal power allocation between the source and relay node. Here, $s_1 = s_2 = -20$ dB, and $s_3 = 0$ dB.

On Fig.8, since all the links have a similar pathloss, the asymptotically optimal allocation gives marginal benefits only for the second cooperation mode. On the other hand, as seen on Fig.9, when the $S \to D$ link is of lower quality, using a relay provides a large gain even if the $S \to R$ link is also weak. Furthermore, if we compare the performances of the second and and third model, we can see a large performance discrepancy when an equal power allocation is used. However, when an asymptotically optimal power allocation is used, both models show similar performances while the third model is more complex to implement in practice. Further analyses have shown that this fact is conditioned on the quality of the $S \to R$ link ; when its quality is low, as in Fig.9, the performance of both models will be close, whereas the third model shows gains when the $S \to R$ link is of superior quality.

# 4 Perspectives

This line of work has many short terms perspectives, as described thereafter:

- The relay power allocation through asymptotic approximations may be directly extended to other metrics. In particular, Annavajjala et al. [2007] only treats the full relay case, the third model of Tab.4, and the second model is rather untreated in the bibliography. A deeper sweep may be required [See Liu et al., 2009] and works on the outage capacity of relays and cooperative networks [Hunter et al., 2006, Laneman et al., 2004].

- It would be interesting to extend the results of relay channels to cooperative multiple access. This joins the work of Hunter et al. [2006], although a different point of view could be to study a *common threshold packet error rate* as a power allocation constraint, along the line of the common achievable rate of information theoretic studies on cooperative multiple access channels.

- This work lacks an analysis of channel coding in slow fading channels. One can numerically obtain asymptotic approximations of the block PER with coding as done by Xi *et al.* [Xi et al., 2011]. Their approach doesn't give analytical solutions, and lacks generality – they consider only one coded block. Going from Mary's PhD thesis analysis of coded PER [Mary, 2008] should be a good starting point. These analyses will involve the Beta function and its inverse, which is available in mathematical packages and usually computed using Newton's method. This could prove a good starting point to evaluate the opportunity of embedded root-finding algorithms for these problems in practical systems.

- The unit-step approximation fits the PER well for large packet size. Preliminary tests tend to show that it also performs well for block coding, and references [Chatzigeorgiou et al., 2008, El Gamal and Hammons, 2001] include similar results for turbo codes. A more comprehensive analysis should include these coding schemes and possibly extend the work of Chatzigeorgiou et al. [2008] to LDPC codes. For those c.d.f. that are not invertible, as in the Rician fading case, this loops back on the previous perspective about embedded root-finding algorithms.

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